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Anomalous Diffusion Limit Induced on a Kinetic Equation

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Using a compactness argument based on the velocity averaging lemma of Golse *et al.*, it is shown that the limiting behavior of a kinetic (linearized BGK) gas model confined between two plates with Maxwell boundary conditions, when the distance between the plates goes to zero, under a suitable anomalous scaling, is diffusive. We do not require the use of central limit theorems as in the method of Börgers *et al.*

KEY WORDS: BGK; nonequilibrium steady states; anomalous diffusion limit.

1. INTRODUCTION

Fluid dynamical descriptions of gases depend on the assumption that the mean free path of a fluid particle (the average distance traveled between collisions) is much smaller compared with the macroscopic length scales of interest. When this assumption breaks down one may abandon fluid dynamics in favor of a kinetic theory of dilute gases like that of the Boltzmann equation. The gas is then described by single particle phase-space densities rather than fluid dynamical variables like the spatial densities of mass momentum and energy. The evolution of these phase-space densities is then governed by kinetic equations.

This paper will treat the case of a simple model of rarefied gas dynamics, in the region located between two infinite flat, parallel surfaces, separated by a small distance h. Gas molecules are assumed to move with constant velocities, and to be reflected according to Maxwell's boundary condition[10] upon impact with the bounding surfaces.

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The purpose of this paper is to show how a rough boundary described by a diffusive boundary condition may generate a diffusion process in a fluid. In contrast with previous results obtained by probabilistic arguments (cf. ref. 7), our proof relies as much as possible on known physical estimates (like conservation laws or the entropy inequality).

The existence of diffusion limits has been demonstrated for other transport problems. Some important examples are contained in the papers of refs. 2, 3, 7, and 12. Reference 12 considers "Arnold's Cat Maps" model as a nice example of a diffusion approximation of a reversible dynamics. In ref. 3, the gas surface interaction was modelled by a boundary condition reminiscent of Maxwell's total accommodation condition; the purpose of this condition was to bias the collision process at the boundary so as to avoid the production of too many particles travelling in directions nearly parallel to the plates. In ref. 7 the genuine Maxwell accommodation condition was treated by probabilistic techniques, and it was shown there that, as the distance between the plates tends to zero, the particles evolve according to an anomalous diffusion process (one calls a diffusion process "*anomalous*" if the mean square displacement grows like a nonlinear function of the time variable (in the long time limit). In the case of a classical diffusion process (such as the classical Brownian motion) the mean square displacement of a particle is proportional to the corresponding time in the long time limit)); In the case considered in ref. 7, the particles travelling in directions nearly parallel to the plates were responsible for the anomalous scaling. The effect of biasing the collision process as was done in ref. 3 eliminated this difficulty, thus leading to a classical diffusion approximation.

In a recent work,[15] F. Golse gives a proof of the result announced in ref. 7 without mentioning probabilitic interpretation of the accommodation boundary condition. His exposition is entirely based on a priori estimates on PDE's. The difference between his approach and ref. 7 is the following: In ref. 7 an explicit representation for the solution of the transport problem is given and analyzed with the help of some limit theorems from probability theory; in ref. 15, F. Golse does neither need any such representation for the solution, nor any probability theory.

The models examined in refs. 7 and 15 are purely non collisional. In this paper, we are taking into account the rare collisions between molecules, which has a regularizing effect for the approximation. We begin with the model of ref. 15, with an additional linearized BGK collision operator, and then apply the method of ref. 15. We prove that, when the time variable is rescaled as $t \to \rho(\varepsilon)$ where ε is the mean free path, i.e., the mean distance of free flight of the particles and $\rho(\varepsilon) = O(\varepsilon)$, the following two different asymptotical regimes can be observed:

— The collisional case, where the frequency of collision $\sigma > 0$ is independent of ε, which forces us to take $\rho(\varepsilon) = \varepsilon^2$, and which is a classical diffusive limit.

— The weakly collisional case, where $\sigma(\varepsilon) = \varepsilon\,|\ln \varepsilon|^\alpha$ with $-1 < \alpha < 0$, which forces us to take $\rho(\varepsilon) = \varepsilon^2\,|\ln \sigma(\varepsilon)|$; in this case, we observe an anomalous diffusive limit, where the mean square displacement per unit of macroscopic time tends to infinity as $|\ln \sigma(\varepsilon)|$.

2. SETTING OF THE PROBLEM AND MAIN RESULTS

Consider a rarefied gas of identical molecules confined between two identical parallel plates; we shall be concerned with the case where the distance between the plates is very small when compared to the size of the plates. If the gas is rarefied enough, one can neglect the effect of intermolecular collisions in the container. The evolution of the density of particles will be modelled by the BGK model and the interaction between gas molecules and the plates, by the "*diffuse reflection*" condition (cf. ref. 10). The density number of particles is denoted by $F \equiv F(t, x, y, z, v_x, v_y, v_z)$ for $t \geqslant 0$, $X = (x, y, z) \in D_h = \mathbb{R} \times \mathbb{R} \times]0, h[$ and $v = (v_x, v_y, v_z) \in \mathbb{R}^3$. In other words, in an infinitesimal volume $dX\,dv$ of the phase space $D_h \times \mathbb{R}^3_v$ at time t one can find $F(t, X, v)\,dX\,dv$ particules.

The evolution of the number density F is governed by the BGK model

$$\partial_t F + v \cdot \nabla_X F = \sigma(M_F - F), \qquad X \in D_h, \quad v \in \mathbb{R}^3 \tag{2.1}$$

Here M_F is a local Maxwellian distribution with the same density, in the velocity space centered at ϱ and u with covariance matrix TI. In the kinetic theory, ϱ represents the (macroscopic) density of the gas, u the bulk velocity of the gas molecules and T the temperature of the gas. In other words

$$M_{(\varrho, u, T)}(v) = \frac{\varrho}{(2\pi T)^{3/2}}\, e^{-|v-u|^2/2T} \tag{2.2}$$

The parameters ϱ, u and T which are functions of t and X are given by the relations:

$$\varrho(t, X) = \int_{\mathbb{R}^3} F(t, X, v)\,dv, \qquad \varrho(t, X)\,u(t, X) = \int_{\mathbb{R}^3} vF(t, X, v)\,dv \tag{2.3}$$

and

$$\varrho(t, X)\,T(t, X) = \tfrac{2}{3} \int_{\mathbb{R}^3} |v - u(t, X)|^2\, F(t, X, v)\,dv \tag{2.4}$$

Finally σ is a positive constant which represents the inverse of the average times of collision, or in other words the relaxation to the Maxwellian position. The boundary condition of diffuse reflection is given by the requirement that on the lower plate $z=0$

$$F(t, x, y, 0, v) = M_{(1,0,T_p)}(v) \frac{\int_{w_z<0} F(t, x, y, 0, w)\,|w_z|\,dw}{\int_{w_z<0} M_{(1,0,T_p)}(w)\,|w_z|\,dw}, \qquad v_z>0 \quad (2.5)$$

and on the upper plate $z=h$

$$F(t, x, y, h, v) = M_{(1,0,T_p)}(v) \frac{\int_{w_z>0} F(t, x, y, 0, w)\,|w_z|\,dw}{\int_{w_z>0} M_{(1,0,T_p)}(w)\,|w_z|\,dw}, \qquad v_z<0 \quad (2.6)$$

The temperature T_p models the temperature field on the plates; evidently the conditions (2.5)–(2.6) are more idealised according to the complexity gas/surface interaction, but are traditional in kinetic theory.

In what follows, we shall assume that the temperature T_p on the plates is constant (in space and time). Furthermore, observe that, without loss of generality, we may assume that $T_p=1$ in order to get asymptotical information about the diffusion coefficient. We assume moreover that the gas is almost at thermic equilibrium state with the plates, of averaging velocity and density. In particular, we assume a negligable non linear effect. We are forced to write the density F in the form of a pertubation of the Maxwellian $M_* = M_{(\varrho_*, 0, T_p)}$, namely

$$F(t, X, v) = M_*(v)(1 + f(t, X, v)) \quad (2.7)$$

where ϱ_* denotes the average value of the macroscopic density of the gas.

Let us write the new equations satisfied by the new unknown f neglecting the non linear terms. First, f satisfies linearized BGK equation around Maxwellian position M_*, namely (cf. ref. 10)

$$\partial_t f + v \cdot \nabla_X f = \sigma(\Pi f - f), \qquad X \in D_h, \quad v \in \mathbb{R}^3 \quad (2.8)$$

In (2.8), Π denotes the orthogonal projection of $L^2(\mathbb{R}^3_v, M_*\,dv)$ on $\text{span}\{1, v_x, v_y, v_z, |v|^2\}$.

For the accommodation boundary conditions that describes the gas/surface interaction in our model, substitute $F = M_*(1+f)$ into the conditions (2.5)–(2.6). Since it is clear that

$$\frac{M_{(1,0,T_p)}}{\int_{w_z<0} M_{(1,0,T_p)}(w)\,|w_z|\,dw} = \frac{M_*}{\int_{w_z<0} M_*(w)\,|w_z|\,dw}$$

these conditions are written in the form

$$f(t, x, y, 0, v) = \frac{\int_{w_z < 0} f(t, x, y, 0, w)\, M_{(1, 0, T_p)}(w)\, |w_z|\, dw}{\int_{w_z > 0} M_{(1, 0, T_p)}(w)\, w_z\, dw}, \qquad v_z > 0 \quad (2.9)$$

and

$$f(t, x, y, h, v) = \frac{\int_{w_z > 0} f(t, x, y, 0, w)\, M_{(1, 0, T_p)}(w)\, w_z\, dw}{\int_{w_z > 0} M_{(1, 0, T_p)}(w)\, w_z\, dw}, \qquad v_z < 0 \quad (2.10)$$

Finally we prescribe an initial condition which is compatible with the expected asymptotic dynamics. This initial condition is of the form:

$$f(0, X, v) = f_0(X, v), \qquad X \in D_h, \quad v \in \mathbb{R}^3 \quad (2.11)$$

Scaling Laws. As the subject of this paper is concerned with the analysis of the dynamical limit of particles in the case where the distance between the plates is very small when compared to the size of the plates, we shall make explicit in this section the condition on different scales of our model. Precisely: it is assumed that the distance h between the "plates" is small when compared to the characteristic length of the horizontal motion (for example, the typical wavelengths in the horizontal Fourier modes of the initial number density). This suggests the rescaling $(x, y) \to (x/\varepsilon, y/\varepsilon)$. Note that, the number of collisions between a typical particle and the plates per unit of unscaled time is $O(1/\varepsilon)$, and as the effect of each of these collisions is to thermalise the particle (according to diffuse reflection relation), it is natural that the limit will be of hydrodynamic type. Hence, in order to observe a horizontal motion at large scale, it is logical to rescale the time variable as $t \to \rho(\varepsilon)$, where $\rho(\varepsilon) = O(\varepsilon)$ as $\varepsilon \to 0$. But so far, we must leave $\rho(\varepsilon)$ unspecified since we suspect an anomalous diffusion regime. The rescaled equation (2.8) then takes the form

$$\rho(\varepsilon)\, \partial_t f_\varepsilon + \varepsilon v_x\, \partial_x f_\varepsilon + \varepsilon v_y\, \partial_y f_\varepsilon + v_z\, \partial_z f_\varepsilon = \sigma(\Pi f_\varepsilon - f_\varepsilon), \qquad X \in D_h, \quad v \in \mathbb{R}^3 \quad (2.12)$$

In what follows we shall consider ε as a small parameter and the frequency of the collision σ can be also small eventually. It is clear that these scalings do not induce any modification in the boundary conditions, namely f_ε still satisfies (2.9)–(2.10). For the initial data, we shall begin with a regular independent density which is independent of the variables v and z. It is of the form

$$f_\varepsilon(0, x, y, z, v) = f_0(x, y), \qquad X \in D_h, \quad v \in \mathbb{R}^3 \quad (2.13)$$

Main Results. We shall begin with the following notation:

Since $M_{(1,0,T_p)}\,dv$ is the probability measure on $\mathbb{R}^3$, we denote by $\langle\phi\rangle$ the average over this measure of any integrable function $\phi = \phi(v)$, namely:

$$\langle\phi\rangle = \int_{\mathbb{R}^3} \phi(v)\, M_{(1,0,T_p)}\, dv$$

Since $(1/h)\,dz\, M_{(1,0,T_p)}\,dv$ is probability measure on $]0,h[\times\mathbb{R}^3$, we denote by $\langle\!\langle\psi\rangle\!\rangle$ the average over this measure of any integrable function $\psi = \psi(z,v)$, namely:

$$\langle\!\langle\psi\rangle\!\rangle = \frac{1}{h}\int_0^h \int_{\mathbb{R}^3} \psi(z,v)\, M_{(1,0,T_p)}\, dz\, dv$$

A simple computation shows that, under these notation, the projection operator Π can be written for any $f \in L^2(\mathbb{R}^3, M_*\, dv)$ as:

$$(\Pi f)(v) = \langle f\rangle + \langle v_x f\rangle \frac{v_x}{T_p} + \langle v_y f\rangle \frac{v_y}{T_p} + \langle v_z f\rangle \frac{v_z}{T_p} + \frac{1}{6}\left\langle\left(\frac{|v|^2}{T_p} - 3\right) f\right\rangle\left(\frac{|v|^2}{T_p} - 3\right) \tag{2.14}$$

The first main result of our paper is the following, as a notational convention, in what follows we use σ to denote σ_h.

Theorem 2.1. Let $\sigma > 0$ fixed and $f_0 \in L^2(\mathbb{R}^2)$. Assume that $\rho(\varepsilon) = \varepsilon^2$. Then

1. for all $\varepsilon > 0$, the system (2.9)–(2.10)–(2.12)–(2.13) has, in the sense of distributions, a unique solution $f_\varepsilon \in L^\infty(\mathbb{R}^+; L^2(D_h \times \mathbb{R}^3, M_*\, dX\, dv))$;
2. when $\varepsilon \to 0$, $f_\varepsilon \to f$ in $w\text{-}L^\infty(\mathbb{R}^+; L^2(D_h \times \mathbb{R}^3, M_*\, dX\, dv))$ and in $L^\infty_{\mathrm{loc}}(\mathbb{R}^+; L^2_{\mathrm{loc}}(D_h \times \mathbb{R}^3, M_*\, dX\, dv))$, where f is a solution of the horizontal diffusion equation

$$\partial_t f - \frac{h}{\sqrt{2\pi}} \ln\left(\frac{1}{\sigma}\right)(\partial_{xx} + \partial_{yy})\, f = 0, \qquad f(0,x,y) = f_0(x,y) \tag{2.15}$$

Note that, Theorem 2.1 deals with the collisional case. Our second result is about the weakly collisional case, that is the case where we have the anomalous diffusion. In this case, the convergence of f_ε to the local equilibrium is not assured by the dissipative character of the collision operator, since the frequency of collision σ goes to 0.

Theorem 2.2. Let $\sigma = \varepsilon\,|\ln \varepsilon|^{\alpha}$, $\alpha \in\,]-1, 0[$. Assume that $\rho(\varepsilon) = \varepsilon^2\,|\ln \sigma(\varepsilon)|$. Then

1. for all $\varepsilon > 0$, the system (2.9)–(2.10)–(2.12)–(2.13) has a unique solution $f_\varepsilon \in L^\infty(\mathbb{R}^+; L^2(D_h \times \mathbb{R}^3, M_*\, dX\, dv))$;

2. when $\varepsilon \to 0$, $f_\varepsilon \to f$ in w-$L^\infty(\mathbb{R}^+; L^2(D_h \times \mathbb{R}^3, M_*\, dX\, dv))$ and in $L^\infty_{\mathrm{loc}}(\mathbb{R}^+; L^2_{\mathrm{loc}}(D_h \times \mathbb{R}^3, M_*\, dX\, dv))$ where f is a solution of diffusion equation. The diffusion coefficient is given by $h/\sqrt{2\pi}$.

3. Moreover, the solution f_ε satisfies the following estimate

$$\left\| \int_0^t f_\varepsilon\, ds - \left\langle\!\left\langle \int_0^t f_\varepsilon\, ds \right\rangle\!\right\rangle \right\|_{L^1(\mathbb{R}^+ \times D_h \times \mathbb{R}^3)} = O(\varepsilon\,|\ln \sigma(\varepsilon)|) + O(\rho(\varepsilon)^{1/2}\,|\ln(\rho(\varepsilon))|^{1/2}) \tag{2.16}$$

locally and uniformly in $t \in [0, T]$.

The reader will refer to the appendix for the notation regarding spaces.

In this work we shall not dwell on the existence and uniqueness proof of a solution for the Cauchy problem (2.9)–(2.10)–(2.12)–(2.13). Let us only mention that the proof can be achieved by a simple contraction type fixed point argument or by standard semigroup methods, as in ref. 3.

This section is concluded by an outline of the remainder of the paper: The following section contains the proof of Theorem 2.1. It relies on the conservation laws for mass. Section 4 is devoted to the study of the "homological equation" which allows us to control the choice of $\sigma(\varepsilon)$ and $\rho(\varepsilon)$. The proof of Theorem 2.2 is carried out in Section 5; it relies on a compactness argument based on the velocity averaging lemma proved in ref. 16.

3. CONSERVATION LAWS AND THE PROOF OF THEOREM 2.1

We start by writing the conservation laws for the system considered in (2.9)–(2.10)–(2.12)–(2.13). The local conservation law of particles' density number is given by

$$\rho(\varepsilon)\,\partial_t\langle f_\varepsilon\rangle + \varepsilon\,\partial_x\langle v_x f_\varepsilon\rangle + \varepsilon\,\partial_y\langle v_y f_\varepsilon\rangle + \partial_z\langle v_z f_\varepsilon\rangle = 0 \tag{3.1}$$

and the local non-increasing linearized entropy law can be written as:

$$\tfrac{1}{2}\rho(\varepsilon)\,\partial_t\langle f_\varepsilon^2\rangle + \tfrac{1}{2}\varepsilon\,\partial_x\langle v_x f_\varepsilon^2\rangle + \tfrac{1}{2}\varepsilon\,\partial_y\langle v_y f_\varepsilon^2\rangle + \tfrac{1}{2}\partial_z\langle v_z f_\varepsilon^2\rangle + \sigma\langle |(I - \Pi)\, f_\varepsilon|^2\rangle = 0 \tag{3.2}$$

It will be necessary to consider the average in z of relations (3.1) and (3.2) above. Averaging over z of (3.1), we obtain:

$$\rho(\varepsilon)\,\partial_t \langle\!\langle f_\varepsilon \rangle\!\rangle + \varepsilon\,\partial_x \langle\!\langle v_x f_\varepsilon \rangle\!\rangle + \varepsilon\,\partial_y \langle\!\langle v_y f_\varepsilon \rangle\!\rangle = 0 \tag{3.3}$$

since the relations (2.9)–(2.10) imply that the net flux of particles on each plate is null; in other words

$$\langle v_z f_\varepsilon(t, x, y, 0, \cdot) \rangle = \langle v_z f_\varepsilon(t, x, y, h, \cdot) \rangle = 0 \tag{3.4}$$

It will be convenient to introduce the following notation: For any function $\phi \in L^1(\mathbb{R}^3, M_{(1,0,T_p)}\, dv)$

$$\begin{aligned} \langle \phi \rangle_+ &= \int_{\mathbb{R}^3} \phi(v)\, M_{(1,0,T_p)} \mathbf{1}_{v_z > 0}\, dv \\ \langle \phi \rangle_- &= \int_{\mathbb{R}^3} \phi(v)\, M_{(1,0,T_p)} \mathbf{1}_{v_z < 0}\, dv \end{aligned} \tag{3.5}$$

Averaging (3.2) in variable z, and taking into account the relations (2.9)–(2.10), yields:

$$\begin{aligned} &\frac{1}{2}\rho(\varepsilon)\,\partial_t \langle\!\langle f_\varepsilon^2 \rangle\!\rangle + \frac{1}{2}\varepsilon\,\partial_x \langle\!\langle v_x f_\varepsilon^2 \rangle\!\rangle + \frac{1}{2}\varepsilon\,\partial_y \langle\!\langle v_y f_\varepsilon^2 \rangle\!\rangle + \sigma \langle\!\langle |(I-\Pi) f_\varepsilon|^2 \rangle\!\rangle \\ &\quad + \frac{1}{2h}\left[\langle v_z f_\varepsilon^2(t, x, y, h, \cdot) \rangle_+ + \left\langle v_z \frac{\langle v_z f_\varepsilon(t, x, y, h, \cdot) \rangle_+^2}{\langle v_z \rangle_+^2} \right\rangle_- \right] \\ &\quad - \frac{1}{2h}\left[\langle v_z f_\varepsilon^2(t, x, y, 0, \cdot) \rangle_- + \left\langle v_z \frac{\langle v_z f_\varepsilon(t, x, y, 0, \cdot) \rangle_-^2}{\langle v_z \rangle_+^2} \right\rangle_+ \right] = 0 \end{aligned} \tag{3.6}$$

We can transform the boundary terms above in such a manner that the sign occurs:

$$\begin{aligned} &\langle v_z f_\varepsilon^2(t, x, y, h, \cdot) \rangle_+ + \left\langle v_z \frac{\langle v_z f_\varepsilon(t, x, y, h, \cdot) \rangle_+^2}{\langle v_z \rangle_+^2} \right\rangle_- \\ &\quad = \langle v_z \rangle_+ \left[\frac{\langle v_z f_\varepsilon^2(t, x, y, h, \cdot) \rangle_+}{\langle v_z \rangle_+} - \left(\frac{\langle v_z f_\varepsilon(t, x, y, h, \cdot) \rangle_+}{\langle v_z \rangle_+} \right)^2 \right] \geqslant 0 \end{aligned} \tag{3.7}$$

and

$$- \langle v_z f_\varepsilon^2(t, x, y, 0, \cdot) \rangle_- - \left\langle \frac{v_z \langle v_z f_\varepsilon(t, x, y, 0, \cdot) \rangle_-^2}{\langle v_z \rangle_+^2} \right\rangle_+$$

$$= \langle v_z \rangle_+ \left[\frac{\langle v_z f_\varepsilon^2(t, x, y, 0, \cdot) \rangle_-}{\langle v_z \rangle_-} - \left(\frac{\langle v_z f_\varepsilon(t, x, y, 0, \cdot) \rangle_-}{\langle v_z \rangle_-} \right)^2 \right] \geqslant 0 \tag{3.8}$$

The fact that the above terms in the inequalities (3.7)–(3.8) are nonnegative is a consequence of Jensen's inequality applied to the probability measure $v_z M_{(1,0,T_p)} \mathbf{1}_{v_z > 0}\, dv / \langle v_z \rangle_+$ and $v_z M_{(1,0,T_p)} \mathbf{1}_{v_z < 0}\, dv / \langle v_z \rangle_-$.

Proof of Theorem 2.1. Integrating the relation (3.6) with respect to the variables $x, y \in \mathbb{R}$ and $t \in [0, T]$, and taking into account the relations (3.7)–(3.8), we get

$$\frac{1}{2} \iint \langle\!\langle f_\varepsilon^2(T, x, y, \cdot, \cdot) \rangle\!\rangle \, dx\, dy$$

$$+ \frac{\sigma}{\rho(\varepsilon)} \iiint \langle\!\langle |(I - \Pi) f_\varepsilon(t, x, y, \cdot, \cdot)|^2 \rangle\!\rangle \, dx\, dy\, dt$$

$$\leqslant \frac{1}{2} \iint \langle\!\langle f_0^2(x, y) \rangle\!\rangle \, dx\, dy \tag{3.9}$$

The relation (3.9) shows that the family (f_ε) is bounded in $L^\infty(\mathbb{R}^+; L^2(D_h \times \mathbb{R}^3; dx\, dy\, dz\, M_{(1,0,T_p)}\, dv))$. This family is then relatively compact in w-$L^\infty(\mathbb{R}^+; L^2(D_h \times \mathbb{R}^3; dx\, dy\, dz\, M_{(1,0,T_p)}\, dv))$. Let f be the limit point of any converging subsequence of (f_ε); the relation (3.9) shows that $f \in Ker(I - \Pi)$. In other words, there exist functions of X and of t, $a(t, X)$, $b_i(t, X)$ for $1 \leqslant i \leqslant 3$ and $c(t, X)$ such that

$$f(t, X, v) = a(t, X) + \sum_{i=1}^{3} b_i(t, X)\, v_i + c(t, X)\, |v^2| \tag{3.10}$$

Moreover, the transform equation (2.12) with $\rho(\varepsilon) = \varepsilon^2$, has the following form

$$\partial_t f_\varepsilon + \frac{1}{\varepsilon} v_x \partial_x f_\varepsilon + \frac{1}{\varepsilon} v_y \partial_y f_\varepsilon + \frac{1}{\varepsilon^2} v_z \partial_z f_\varepsilon = \frac{\sigma}{\varepsilon^2} (\Pi f_\varepsilon - f_\varepsilon), \qquad X \in D_h, \quad v \in \mathbb{R}^3$$

Since Π is the orthogonal projection on the kernel of $v_z \partial_z f$, integrating the last equation over all variables and multiplying by ε^2, show that f_ε is independent of the variable z; hence

$$v_z \partial_z f = 0 \tag{3.11}$$

This result, together with (3.10) show that the functions $a(t, X)$, $b_i(t, X)$ for $1 \leqslant i \leqslant 3$ and $c(t, X)$ are in fact independent of the variable z. Finally, relations (3.6)–(3.7) show that

$$\left[\frac{\langle v_z f^2(t, x, y \cdot) \rangle_+}{\langle v_z \rangle_+} - \left(\frac{\langle v_z f(t, x, y \cdot) \rangle_+}{\langle v_z \rangle_+} \right)^2 \right] = 0 \tag{3.12}$$

and

$$\left[\frac{\langle v_z f^2(t, x, y, \cdot) \rangle_-}{\langle v_z \rangle_-} - \left(\frac{\langle v_z f(t, x, y, \cdot) \rangle_-}{\langle v_z \rangle_-} \right)^2 \right] = 0 \tag{3.13}$$

Since these two terms are the variances of $f(t, x, y, \cdot)$ for the probabilities $v_z M_{(1, 0, T_p)} \mathbf{1}_{v_z > 0}\, dv / \langle v_z \rangle_+$ and $v_z M_{(1, 0, T_p)} \mathbf{1}_{v_z < 0}\, dv / \langle v_z \rangle_-$ respectively, we deduce from (3.12)–(3.13) that $f(t, x, y, \cdot)\, \mathbf{1}_{v_z > 0}$ and $f(t, x, y, \cdot)\, \mathbf{1}_{v_z < 0}$ are independent of v, which implies, because of (3.10), that $f(t, x, y, \cdot)$ is independent of v, namely the functions $b_i(t, X)$ for $1 \leqslant i \leqslant 3$ and $c(t, X)$ are identically zero.

Now, let $\mathscr{L}$ be the unbounded operator on $L^2([0, h] \times \mathbb{R}^3; dz\, M_{(1, 0, T_p)}\, dv)$ with the domain $\mathscr{D}(\mathscr{L})$ given by

$$\mathscr{D}(\mathscr{L}) = \Big\{ f \in L^2([0, h] \times \mathbb{R}^3; dz\, M_{(1, 0, T_p)}\, dv) \mid$$

$$v_z \partial_z f \in L^2([0, h] \times \mathbb{R}^3; dz\, M_{(1, 0, T_p)}\, dv)$$

$$\text{and } f(0, v) = \frac{\langle v_z f(0, \cdot) \rangle_-}{\langle v_z \rangle_-}, \ v_z > 0;$$

$$f(h, v) = \frac{\langle v_z f(h, \cdot) \rangle_+}{\langle v_z \rangle_+}, \ v_z < 0 \Big\}$$

and defined by

$$\mathscr{L} f = v_z \partial_z f + \sigma(f - \Pi f) \tag{3.14}$$

Also, define the unbounded operator $\mathscr{L}^*$ on $L^2([0,h]\times\mathbb{R}^3;\, dz\, M_{(1,0,T_p)}\, dv)$ with the domain

$$\mathscr{D}(\mathscr{L}^*) = \Big\{ f \in L^2([0,h]\times\mathbb{R}^3;\, dz\, M_{(1,0,T_p)}\, dv) \mid$$
$$v_z\,\partial_z f \in L^2([0,h]\times\mathbb{R}^3;\, dz\, M\, dv)$$
$$\text{and } f(0,v) = \frac{\langle v_z f(0,\cdot)\rangle_+}{\langle v_z\rangle_+},\; v_z < 0;$$
$$f(h,v) = \frac{\langle v_z f(h,\cdot)\rangle_-}{\langle v_z\rangle_-},\; v_z > 0 \Big\}$$

and defined by

$$\mathscr{L}^* f = -v_z\,\partial_z f + \sigma(f - \Pi f) \tag{3.15}$$

First, we start by the observation that, if g_1 and g_2 are smooth functions on $[0,h]\times\mathbb{R}^3$, then

$$\begin{aligned}\langle\!\langle g_2 \mathscr{L} g_1\rangle\!\rangle - \langle\!\langle g_1 \mathscr{L}^* g_2\rangle\!\rangle &= \langle\!\langle v_z\,\partial_z(g_1 g_2)\rangle\!\rangle \\ &= \langle v_z g_1 g_2(h,\cdot)\rangle - \langle v_z g_1 g_2(0,\cdot)\rangle\end{aligned}$$

Note that, this equality still holds if $g_1, g_2, v_z\,\partial_z g_1$ and $v_z\,\partial_z g_2 \in L^2([0,h]\times\mathbb{R}^3)$.

Let $g_1 \in \mathscr{D}(\mathscr{L})$ and $g_2 \in \mathscr{D}(\mathscr{L}^*)$ as defined above; then

$$\langle v_z g_1(h,\cdot)\rangle_+ \frac{\langle v_z g_2(h,\cdot)\rangle_-}{\langle v_z\rangle_-} + \langle v_z g_2(h,\cdot)\rangle_- \frac{\langle v_z g_1(h,\cdot)\rangle_+}{\langle v_z\rangle_+} = 0 \tag{3.16}$$

Similarly

$$-\langle v_z g_1(0,\cdot)\rangle_- \frac{\langle v_z g_2(0,\cdot)\rangle_+}{\langle v_z\rangle_+} + \langle v_z g_2(0,\cdot)\rangle_+ \frac{\langle v_z g_1(0,\cdot)\rangle_-}{\langle v_z\rangle_-} = 0 \tag{3.17}$$

Hence, if $g_1 \in \mathscr{D}(\mathscr{L})$ and $g_2 \in \mathscr{D}(\mathscr{L}^*)$, we have

$$\langle\!\langle g_2 \mathscr{L} g_1\rangle\!\rangle = \langle\!\langle g_1 \mathscr{L}^* g_2\rangle\!\rangle$$

This shows that $\mathscr{L}^*$ is included in the adjoint of $\mathscr{L}$.

We will later prove the following result (cf. Section 4):

Lemma 3.1. There exist functions $\beta_x \in \mathscr{D}(\mathscr{L}^*)$ and $\beta_y \in \mathscr{D}(\mathscr{L}^*)$ such that

$$\mathscr{L}^*\beta_x = v_x, \qquad \mathscr{L}^*\beta_y = v_y$$

We now finish the proof of Theorem 2.1. The relation (3.3) takes the form

$$\partial_t \langle\!\langle f_\varepsilon \rangle\!\rangle + \frac{\varepsilon}{\rho(\varepsilon)} \partial_x \langle\!\langle \mathscr{L}^*\beta_x f_\varepsilon \rangle\!\rangle + \frac{\varepsilon}{\rho(\varepsilon)} \partial_y \langle\!\langle \mathscr{L}^*\beta_y f_\varepsilon \rangle\!\rangle = 0$$

or, using relations (3.16)–(3.17),

$$\partial_t \langle\!\langle f_\varepsilon \rangle\!\rangle + \frac{\varepsilon}{\rho(\varepsilon)} \partial_x \langle\!\langle \beta_x \mathscr{L} f_\varepsilon \rangle\!\rangle + \frac{\varepsilon}{\rho(\varepsilon)} \partial_y \langle\!\langle \beta_y \mathscr{L} f_\varepsilon \rangle\!\rangle = 0 \tag{3.18}$$

But the equation (2.12) shows that

$$\mathscr{L} f_\varepsilon = -\varepsilon v_x \partial_x f_\varepsilon - \varepsilon v_y \partial_y f_\varepsilon - \rho(\varepsilon) \partial_t f_\varepsilon$$

so (3.18) can be rewritten as

$$\partial_t \langle\!\langle f_\varepsilon \rangle\!\rangle - \frac{\varepsilon^2}{\rho(\varepsilon)} \partial_x \langle\!\langle v_x \beta_x \partial_x f_\varepsilon \rangle\!\rangle - \frac{\varepsilon^2}{\rho(\varepsilon)} \partial_y \langle\!\langle v_y \beta_y \partial_y f_\varepsilon \rangle\!\rangle$$
$$= \varepsilon\, \partial_x \partial_t \langle\!\langle \beta_x f_\varepsilon \rangle\!\rangle + \varepsilon\, \partial_y \partial_t \langle\!\langle \beta_y f_\varepsilon \rangle\!\rangle$$

Since f is a w-$L^\infty(\mathbb{R}^+; L^2(D_h \times \mathbb{R}^3; dx\, dy\, dz\, M_{(1,0,T_p)}\, dv))$ limit point of any converging subsequence of (f_ε), we deduce from (3.19) that, choosing

$$\rho(\varepsilon) = \varepsilon^2 \tag{3.20}$$

yields

$$\partial_t \langle\!\langle f \rangle\!\rangle - \partial_x \langle\!\langle v_x \beta_x \rangle\!\rangle\, \partial_x f - \partial_y \langle\!\langle v_y \beta_y \rangle\!\rangle\, \partial_y f = 0 \tag{3.21}$$

We also have the relation

$$f(0, x, y) = f_0(x, y) \tag{3.22}$$

on account of (2.16) and the fact that the family $\partial_t \langle\!\langle f_\varepsilon \rangle\!\rangle$ is bounded in $L^\infty(\mathbb{R}^+; H^{-2}(\mathbb{R}^2))$, from (3.19).

Finally

$$\begin{aligned}\langle\!\langle v_x\beta_x\rangle\!\rangle &= \sigma\langle\!\langle |(I-\Pi)\,\beta_x|^2\rangle\!\rangle - \langle\!\langle v_z\,\partial_z(\beta_x^2)\rangle\!\rangle \\ &= \sigma\langle\!\langle |(I-\Pi)\,\beta_x|^2\rangle\!\rangle \geqslant 0 \end{aligned} \tag{3.23}$$

since the boundary conditions on β_x impose that the net flux of β_x on each plate is zero. In fact, the inequality (3.23) is strict: otherwise, we would have $\beta_x \in Ker(I-\Pi)$, in other words, β_x would be in the form of the right side of (3.10) which contradicts the fact that $\mathscr{L}^*\beta_x = 0$. So, $\langle\!\langle v_x\beta_x\rangle\!\rangle > 0$ as well as $\langle\!\langle v_y\beta_y\rangle\!\rangle > 0$. The uniqueness of solutions of the Cauchy problem (3.21)–(3.22) and the weak relative compactness of the sequence (f_ε) guarantees that the family f_ε converges to f in $w\text{-}L^\infty(\mathbb{R}^+;\ L^2(D_h\times\mathbb{R}^3;\ dx\,dy\,dz\,M_{(1,0,T_p)}\,dv))$. This completes the proof of Theorem 2.1. ∎

4. THE HOMOLOGICAL EQUATION AND THE PROOF OF LEMMA 3.1

In this section, we introduce and study the "homological equation." The terminology homological equation is borrowed from the averaging theory of dynamical systems, see ref. 1. The idea of introducing this equation has proved to be fruitful in many questions related to kinetic equations or dynamical systems (see ref. 13). The homological equation is given by

$$\begin{cases} -v_z\,\partial_z\beta_x + \sigma(I-\Pi)\,\beta_x = v_x, & z\in\,]0,h[,\quad v_z\in\mathbb{R},\quad v_x\in\mathbb{R} \\ \beta_x(0,v_x,v_z) = \dfrac{\int_{w_z>0} w_z\beta_x(0,w)\,M_{(1,0,T_p)}(w)\,dw}{\int_{w_z>0} w_z M_{(1,0,T_p)}(w)\,dw}, & v_x\in\mathbb{R},\quad v_z<0 \\ \beta_x(h,v_x,v_z) = \dfrac{\int_{w_z<0} |w_z|\,\beta_x(h,w)\,M_{(1,0,T_p)}(w)\,dw}{\int_{w_z>0} w_z M_{(1,0,T_p)}(w)\,dw}, & v_x\in\mathbb{R},\quad v_z>0 \end{cases} \tag{4.1}$$

Lemma 4.1. For any $\sigma>0$ and $T_p>0$, the system (4.1) has a unique solution $\beta_x\in\mathscr{D}(\mathscr{L}^*)$ with the average $\langle\!\langle\beta_x^\sigma\rangle\!\rangle=0$, which is given by the form $\beta_x^\sigma(z,v)=v_x\psi_\sigma(z,v_z)$. Moreover, one has the following asymptotic equivalent

$$\langle\!\langle v_x\beta_x^\sigma\rangle\!\rangle \sim \frac{h}{\sqrt{2\pi}}\ln\left(\frac{1}{\sigma}\right) \tag{4.2}$$

with

$$\|\langle |\beta_x^\sigma| \rangle\|_{L^\infty([0,h])} = O(|\ln \sigma|), \qquad \langle\langle |\beta_x^\sigma|^2 \rangle\rangle = O(1/\sigma) \tag{4.3}$$

as $\sigma \to 0^+$.

Observe that Lemma 4.1 implies clearly Lemma 3.1 of the previous section because of the symmetry between the variables v_x and v_y.

As mentioned in the introduction, we shall assume without loss of generality that $T_p = 1$, and we denote $M = M_{(1,0,1)}$. So the projector operation Π can be written as

$$\Pi f = \langle f \rangle + \sum_{i=1}^{3} \langle f_i v_i \rangle \, v_i + \tfrac{1}{6} \langle (|v|^2 - 3) f \rangle (|v|^2 - 3) \tag{4.4}$$

Proof of Lemma 4.1. The uniqueness is proved as follows. Let $b \equiv b(z, v) \in \mathscr{D}(\mathscr{L}^*)$ be a solution of the homogeneous problem (4.1) (in other words, the second side v_x is replaced by 0). Multiplying system (4.1) by b and integrating over all variables (using computations in (3.6) and (3.7)) leads to

$$\begin{aligned} 0 &= \sigma \langle\langle |(I - \Pi)\, b|^2 \rangle\rangle \\ &+ \frac{1}{2h} \langle v_z \rangle_+ \left[\frac{\langle v_z b^2(0, \cdot) \rangle_+}{\langle v_z \rangle_+} - \left(\frac{\langle v_z b(0, \cdot) \rangle_+}{\langle v_z \rangle_+} \right)^2 \right] \\ &+ \frac{1}{2h} \langle v_z \rangle_+ \left[\frac{\langle v_z b^2(h, \cdot) \rangle_-}{\langle v_z \rangle_-} - \left(\frac{\langle v_z b(h, \cdot) \rangle_-}{\langle v_z \rangle_-} \right)^2 \right] \end{aligned} \tag{4.5}$$

The right hand side of (4.5) is the sum of three nonnegative or zero terms; then (4.5) implies that each of these terms is zero. We then deduce first that $b \in Ker(I - \Pi)$, which implies, using the homogeneous problem (4.1) (i.e., with second side zero) that b is independent of z. The same argument as in Section 3 (see formula (3.8) and the following section) shows that b is reduced to a constant. But the condition $\langle\langle b \rangle\rangle = 0$ implies $b = 0$. This proves the uniqueness part of the lemma.

Now, we prove the existence of solutions and the announced asymptotic equivalent (4.2)–(4.3). Consider the auxilary problem

$$\begin{cases} \sigma\psi_\sigma - v_z\, \partial_z \psi_\sigma = 1 + \sigma\widetilde{\psi}_\sigma(z) \\ \psi_\sigma(0, v_z) = 0, \qquad v_z < 0 \\ \psi_\sigma(h, v_z) = 0, \qquad v_z > 0 \end{cases} \tag{4.6}$$

where

$$\widetilde{\psi}_\sigma(z) = \int_{\mathbb{R}} \psi_\sigma(z, v_z)\, e^{-v_z^2/2}\, \frac{dv_z}{\sqrt{2\pi}} \tag{4.7}$$

We now write the integral expressions for (4.6). For $v_z < 0$,

$$\begin{aligned} \psi_\sigma(z, v_z) &= -\int_0^z e^{(\sigma/v_z)(z-t)}\, \frac{1}{v_z}\,(1+\sigma\widetilde{\psi}_\sigma(t))\, dt \\ &= \int_0^z e^{-(\sigma/|v_z|)(z-t)}\, \frac{1}{|v_z|}\,(1+\sigma\widetilde{\psi}(t))\, dt \end{aligned} \tag{4.8}$$

and for $v_z > 0$,

$$\psi_\sigma(z, v_z) = \int_z^h e^{-(\sigma/v_z)(t-z)}\, \frac{1}{v_z}\,(1+\sigma\widetilde{\psi}_\sigma(t))\, dt \tag{4.9}$$

We then deduce a unique integral equation for the unknown $\widetilde{\psi}_\sigma$ (which is known as Peierls Equation. This terminology comes from neutron transport theory):

$$\begin{aligned} \widetilde{\psi}_\sigma(z) &= \int_0^z (1+\sigma\widetilde{\psi}_\sigma(t))\, dt \int_{v_z<0} e^{-(\sigma/|v_z|)(z-t)}\, \frac{1}{|v_z|}\, e^{-v_z^2/2}\, \frac{dv_z}{\sqrt{2\pi}} \\ &\quad + \int_z^h (1+\sigma\widetilde{\psi}_\sigma(t))\, dt \int_{v_z>0} e^{-(\sigma/v_z)(t-z)}\, \frac{1}{v_z}\, e^{-v_z^2/2}\, \frac{dv_z}{\sqrt{2\pi}} \end{aligned} \tag{4.10}$$

Introduce then the function

$$J(z) = \int_{v_z>0} e^{-(1/v_z)\,z - v_z^2/2}\, \frac{1}{v_z}\, \frac{dv_z}{\sqrt{2\pi}} \tag{4.11}$$

the relation (4.10) can be evidently written as

$$\widetilde{\psi}_\sigma(z) = \int_0^h (1+\sigma\widetilde{\psi}(t))\, J(\sigma|z-t|)\, dt \tag{4.12}$$

We shall study the behaviour of J near of zero. Set $u = 1/v_z$

$$J(z) = \int_0^\infty e^{-zu - 1/u^2}\, \frac{du}{\sqrt{2\pi}\, u} \geqslant 0 \tag{4.13}$$

At first, it is clear that

$$\left| \int_0^1 e^{-zu - 1/u^2} \frac{du}{u} \right| = O(1) \tag{4.14}$$

and then

$$J(z) = O(1) + \frac{1}{\sqrt{2\pi}} \int_1^\infty e^{-zu - 1/u^2} \frac{du}{u} \tag{4.15}$$

Now

$$\int_1^\infty e^{-zu - 1/u^2} \frac{du}{u} \leqslant \int_1^\infty e^{-zu} \frac{du}{u} = \int_z^\infty e^{u} \frac{du}{u} = O(|\ln z|) \tag{4.16}$$

as $z \to 0^+$. Therefore

$$J(z) = O(|\ln z|) \qquad \text{as} \quad z \to 0^+ \tag{4.17}$$

Then, let K_σ be the integral operator defined by

$$(K_\sigma f)(z) = \sigma \int_0^h J(\sigma\, |z-t|)\, f(t)\, dt, \qquad 0 \leqslant t \leqslant h \tag{4.18}$$

The relation (4.17) shows that

$$0 \leqslant \sigma J(\sigma\, |z-t|) \leqslant C \sqrt{\sigma}\, |z-t|^{-1/2}$$

so K_σ is a bounded operator on $L^\infty([0,h])$ with norm $\|K_\sigma\| = O(\sqrt{\sigma})$. Indeed

$$\begin{aligned} \sup_{z \in [0,h]} \int_0^h |f(t)|\, |z-t|^{-1/2}\, dt &\leqslant \|f_\sigma\|_{L^\infty([0,h])} \sup_{z \in [0,h]} \int_0^h |z-t|^{-1/2}\, dt \\ &= 2\, \|f_\sigma\|_{L^\infty([0,h])} \sup_{z \in [0,h]} (\sqrt{z} + \sqrt{h-z}) \\ &\leqslant 4\sqrt{h}\, \|f_\sigma\|_{L^\infty([0,h])} \end{aligned} \tag{4.19}$$

There exists $\sigma_0 > 0$ such that, for any $0 < \sigma < \sigma_0$, $\|K_\sigma\| < 1/2$, we have

$$\tilde{\psi}_\sigma = \sum_{n \geqslant 0} K_\sigma^n(f_\sigma) \tag{4.20}$$

where

$$f_\sigma(z) = \int_0^h J(\sigma\,|z-t|)\,dt, \qquad 0 \leqslant z \leqslant h \tag{4.21}$$

Consequently, for any $0 < \sigma < \sigma_0$,

$$\begin{aligned}\|\tilde{\psi}_\sigma\|_{L^\infty([0,h])} &\leqslant \|f_\sigma\|_{L^\infty([0,h])} \sum_{n \geqslant 0} \|K_\sigma^n\| \\ &\leqslant 2\,\|f_\sigma\|_{L^\infty([0,h])}\end{aligned} \tag{4.22}$$

and (4.17) shows that

$$\|f_\sigma\|_{L^\infty([0,h])} = O(|\ln \sigma|) \tag{4.23}$$

We deduce from (4.22) and (4.21) that, for $0 < \sigma < \sigma_0$, the problem (4.12) has a solution $\tilde{\psi}_\sigma \in L^\infty([0,h])$ satisfying

$$\|\tilde{\psi}_\sigma\|_{L^\infty([0,h])} = O(|\ln \sigma|) \tag{4.24}$$

Now we estimate $\psi_\sigma^2(z, v_z)$. There exists a constant $C > 0$ such that, for $0 < \sigma < \sigma_0$, we have

$$\begin{aligned}\psi_\sigma^2(z, v_z) &= \left(\int_z^h e^{-(\sigma/v_z)(z-t)} \frac{1}{v_z} (1 + \sigma\tilde{\psi}(t))\,dt \right)^2 \\ &\leqslant (1 + C\sigma\,|\ln \sigma|)^2 \left(\int_z^h e^{-(\sigma/v_z)(z-t)} \frac{dt}{v_z} \right)^2\end{aligned} \tag{4.25}$$

We have also an explicit formula for the integral appearing in the right hand side of (4.24)

$$\int_z^h e^{-(\sigma/v_z)(z-t)} \frac{dt}{v_z} = \frac{1}{\sigma} \int_0^{h-z} e^{-(\sigma/v_z)\,\xi}\,\sigma\,\frac{dz}{v_z} = \frac{1}{\sigma}\,(1 - e^{-(\sigma/v_z)(h-z)})$$

Consequently

$$\begin{aligned}&\int_0^h \int_{\mathbb{R}} \psi_\sigma(z, v_z)^2\, e^{-v_z^2/2}\,dv_z\,dz \\ &\qquad \leqslant (1 + C\sigma\,|\ln \sigma|)^2 \int_0^h \int_0^\infty \frac{1}{\sigma^2}\,(1 - e^{-(\sigma/v_z)(h-z)})^2\, e^{-v_z^2/2}\,dz\,dv_z\end{aligned} \tag{4.26}$$

The change of variables $\theta=\sigma(h-z)/v_z$ leads to

$$\int_0^h \int_0^\infty \frac{1}{\sigma^2}\,(1-e^{-(\sigma/v_z)(h-z)})^2\, e^{-v_z^2/2}\, dz\, dv_z$$

$$=\frac{1}{\sigma}\int_0^h \left(\int_0^\infty (1-e^{\theta})^2\, e^{-\sigma^2(h-z)^2/2}\,\frac{d\theta}{\theta^2}\right)(h-z)\, dv_z$$

As $0\leqslant e^{-\sigma^2(h-z)^2/2\theta^2}\leqslant 1$, the dominated convergence theorem shows that, for $\sigma\to 0$,

$$\int_0^h \int_0^\infty \frac{1}{\sigma}\,(1-e^{-(\sigma/v_z)(h-z)})^2\, e^{-v_z^2/2}\, dz\, dv_z \to \int_0^h \left(\int_0^\infty (1-e^{\theta})^2\,\frac{d\theta}{\theta^2}\right)(h-z)\, dv_z$$

More precisely

$$\int_0^h \int_0^\infty \frac{1}{\sigma^2}\,(1-e^{-(\sigma/v_z)(h-z)})^2\, e^{-v_z^2/2}\, dz\, dv_z \sim \frac{1}{\sigma}\,\frac{h^2}{2}\int_0^\infty (1-e^{\theta})^2\,\frac{d\theta}{\theta^2} \tag{4.27}$$

We have thus proved, for $\sigma>0$ small enough, the existence of a solution ψ_σ of (4.6) satisfying (4.23) as

$$\int_0^h \int_{\mathbb{R}} \psi_\sigma(z, v_z)^2\, e^{-v_z^2/2}\, dz\, dv_z = O(1/\sigma) \tag{4.28}$$

from (4.25)–(4.26). The existence of β_x^σ satisfying (4.2)–(4.3) follows immediately. ▮

5. CONVERGENCE TO THE LOCAL EQUILIBRIUM AND THE PROOF OF THEOREM 2.2

The proof of Theorem 2.2 followed closely the proof of Theorem 2.1 with some extra technical complications arising from the fact that the convergence of f_ε to the local equilibrium is not assured by the dissipative character of the collision operator, since the frequency of collision σ goes to 0, as already mentionned in Section 2. On the contrary, the convergence to local equilibrum is a consequence of the diffusion reflection on the surfaces and the fact that the distance between the plates is very small. We then followed the arguments of the Section 3 by indicating the necessary modifications.

The continuity equation (3.18) can be stated in the form

$$\partial_t \langle\!\langle f_\varepsilon \rangle\!\rangle - \frac{\varepsilon^2}{\rho(\varepsilon)} \partial_x^2 \langle\!\langle v_x \beta_x f_\varepsilon \rangle\!\rangle - \frac{\varepsilon^2}{\rho(\varepsilon)} \partial_y^2 \langle\!\langle v_y \beta_y f_\varepsilon \rangle\!\rangle$$
$$= \varepsilon\, \partial_x \partial_t \langle\!\langle \beta_x f_\varepsilon \rangle\!\rangle + \varepsilon\, \partial_y \partial_t \langle\!\langle \beta_y f_\varepsilon \rangle\!\rangle \tag{5.1}$$

To start with, we must write a relation connecting σ to ε expressing the fact that the right hand side of (5.1) tends to 0 with ε. Therefore

$$|\langle\!\langle \beta_x f_\varepsilon \rangle\!\rangle| \leqslant \langle\!\langle |\beta_x^\sigma| \rangle\!\rangle \sup_{0<\varepsilon<1} \|f_\varepsilon\|_{L^\infty(\mathbb{R}^+ \times D_h \times \mathbb{R}^3)}$$
$$\leqslant O(|\ln \sigma(\varepsilon)|) \sup_{0<\varepsilon<1} \|f_\varepsilon\|_{L^\infty(\mathbb{R}^+ \times D_h \times \mathbb{R}^3)} \tag{5.2}$$

and

$$|\langle\!\langle \beta_y f_\varepsilon \rangle\!\rangle| \leqslant \langle\!\langle |\beta_y^\sigma| \rangle\!\rangle \sup_{0<\varepsilon<1} \|f_\varepsilon\|_{L^\infty(\mathbb{R}^+ \times D_h \times \mathbb{R}^3)}$$
$$\leqslant O(|\ln \sigma(\varepsilon)|) \sup_{0<\varepsilon<1} \|f_\varepsilon\|_{L^\infty(\mathbb{R}^+ \times D_h \times \mathbb{R}^3)} \tag{5.3}$$

First, we shall show that

$$\sup_{0<\varepsilon<1} \|f_\varepsilon\|_{L^\infty(\mathbb{R}^+ \times D_h \times \mathbb{R}^3)} < +\infty \tag{5.4}$$

In order to prove (5.4), we write Eq. (2.12) in the form

$$\partial_t f_\varepsilon + \left(\frac{\varepsilon}{\rho(\varepsilon)} v_x \partial_x + \frac{\varepsilon}{\rho(\varepsilon)} v_y \partial_y + \frac{1}{\rho(\varepsilon)} v_z \partial_z \right) f_\varepsilon = \frac{\sigma(\varepsilon)}{\rho(\varepsilon)} (\Pi - I)\, f_\varepsilon \tag{5.5}$$

and set

$$A = \frac{\varepsilon}{\rho(\varepsilon)} v_x \partial_x + \frac{\varepsilon}{\rho(\varepsilon)} v_y \partial_y + \frac{1}{\rho(\varepsilon)} v_z \partial_z, \qquad B = \Pi - I, \text{ and } S(t) = e^{-tA} \tag{5.6}$$

Now, using the following equivalent integral representation of (5.5) and (2.13) given by Duhamel's principle, we get:

$$f_\varepsilon(t, ., ., .) = S_\varepsilon(t)\, f_0 + \int_0^t S_\varepsilon(t-s) \frac{\sigma(\varepsilon)}{\rho(\varepsilon)} B f_\varepsilon(s)\, ds \tag{5.7}$$

Observe that the formula

$$e^{-t(A+B)} - e^{-tA} = \int_0^t \frac{d}{ds}\,(e^{-sA} e^{-(t-s)(A+B)})\,ds$$

$$= -\int_0^t e^{-sA} B e^{-(t-s)(A+B)}\,ds \tag{5.8}$$

is true for any bounded operators A and B. This formula also remains true for semi-groups. Since $S(t)$ is a contraction in $L^\infty([0,h]\times\mathbb{R}^3)$, we have

$$e^{-t(A-(\sigma(\varepsilon)/\rho(\varepsilon))\,B)} = e^{-tA} + \int_0^t e^{-(t-s)\,A}\,\frac{\sigma(\varepsilon)}{\rho(\varepsilon)}\,Be^{-s(A-(\sigma(\varepsilon)/\rho(\varepsilon))\,B)}\,ds \tag{5.9}$$

from which it follows that

$$\|e^{-t(A-(\sigma(\varepsilon)/\rho(\varepsilon))\,B)}\|_{L^\infty}$$

$$\leqslant C\,\|e^{-tA}\|_{L^\infty} + \int_0^t \|e^{-(t-s)\,A}\|_{L^\infty}\,\frac{\sigma(\varepsilon)}{\rho(\varepsilon)}\,\|B\|_{L^\infty}\,\|e^{-s(A-(\sigma(\varepsilon)/\rho(\varepsilon))\,B)}\|_{L^\infty}\,ds \tag{5.10}$$

Note that

$$\|e^{-tA}\|_{L^\infty} \leqslant 1, \qquad \text{and} \qquad \|e^{-(t-s)\,A}\|_{L^\infty} \leqslant 1$$

therefore

$$\|e^{-t(A-(\sigma(\varepsilon)/\rho(\varepsilon))\,B)}\|_{L^\infty} \leqslant 1 + \frac{\sigma(\varepsilon)}{\rho(\varepsilon)}\,\|B\|_{L^\infty}\int_0^t e^{-s(A-\sigma((\varepsilon)/\rho(\varepsilon))\,B)}\,ds \tag{5.11}$$

From Gronwall's lemma we obtain

$$\|e^{-t(A-(\sigma(\varepsilon)/\rho(\varepsilon))\,B)}\|_{L^\infty} \leqslant e^{(\sigma(\varepsilon)/\rho(\varepsilon))\,\|B\|\,t}, \qquad \text{on} \quad [0, \rho(\varepsilon)/\sigma(\varepsilon)] \tag{5.12}$$

More precisely, setting $M = e^{(\sigma(\varepsilon)/\rho(\varepsilon))\,\|B\|\,t}$, we have $\|f_\varepsilon\|_{L^\infty} \leqslant M\,\|f_0\|_{L^\infty}$. Arguing as above, changing f_ε into $-f_\varepsilon$ and writing the boundary limit of $-f_0 \leqslant 0$, we obtain $f_\varepsilon \geqslant 0$, a.e. on $\mathbb{R}^+\times\mathbb{R}^2\times[0,h]\times\mathbb{R}^3$; which completes the proof of the estimate (5.4).

The choice of $\sigma(\epsilon)$ and $\rho(\epsilon)$. The right hand side of (5.1) converges to 0 in the sense of distributions, provided

$$\varepsilon\,|\ln\sigma(\varepsilon)| \to 0, \qquad \text{as} \quad \varepsilon \to 0 \tag{5.13}$$

Hence, if $\sigma(\varepsilon)$ satisfy (5.13) and

$$\frac{\varepsilon^2}{\rho(\varepsilon)}\,|\ln\sigma| \to \kappa, \qquad \text{as} \quad \varepsilon \to 0$$

where κ is some constant, we have

$$\partial_t \langle\!\langle f_\varepsilon \rangle\!\rangle - \frac{h}{\sqrt{2\pi}}\,(\partial_x^2 \langle\!\langle f_\varepsilon \rangle\!\rangle + \partial_y^2 \langle\!\langle f_\varepsilon \rangle\!\rangle) \to 0 \tag{5.14}$$

in $H^{-3}_{\text{loc}}(\mathbb{R}^+ \times \mathbb{R}^2)$. Indeed, recall the continuity equation in the form

$$\begin{aligned}
\partial_t \langle\!\langle f_\varepsilon \rangle\!\rangle &- \frac{\varepsilon^2}{\rho(\varepsilon)}\,\partial_x^2 \langle\!\langle v_x \beta_x(z, v) \langle f_\varepsilon \rangle (t, x, y, z) \rangle\!\rangle \\
&- \frac{\varepsilon^2}{\rho(\varepsilon)}\,\partial_y^2 \langle\!\langle v_y \beta_y(z, v) \langle f_\varepsilon \rangle (t, x, y, z) \rangle\!\rangle \\
&= \varepsilon\,\partial_x \partial_t \langle\!\langle \beta_x f_\varepsilon \rangle\!\rangle + \varepsilon\,\partial_y \partial_t \langle\!\langle \beta_y f_\varepsilon \rangle\!\rangle \\
&+ \frac{\varepsilon^2}{\rho(\varepsilon)}\,\partial_x^2 \langle\!\langle v_x \beta_x(z, v)(f_\varepsilon - \langle f_\varepsilon \rangle) \rangle\!\rangle \\
&+ \frac{\varepsilon^2}{\rho(\varepsilon)}\,\partial_y^2 \langle\!\langle v_y \beta_y(z, v)(f_\varepsilon - \langle f_\varepsilon \rangle) \rangle\!\rangle
\end{aligned}$$

The maximum principle allows us to write:

$$\|f_\varepsilon\|_{L^\infty} \leqslant C \tag{5.15}$$

Hence

$$\varepsilon \langle\!\langle \beta_x f_\varepsilon \rangle\!\rangle \leqslant \varepsilon \langle\!\langle |\beta_x| \rangle\!\rangle\; C = O(\varepsilon\,|\ln\sigma|) \tag{5.16}$$

and

$$\varepsilon \langle\!\langle \beta_y f_\varepsilon \rangle\!\rangle \leqslant \varepsilon \langle\!\langle |\beta_y| \rangle\!\rangle\; C = O(\varepsilon\,|\ln\sigma|) \tag{5.17}$$

On the other hand

$$\|v_x \beta_x^\sigma\|_{L^2(t, x, y, v)} = O\left(\frac{1}{\sqrt{\sigma}}\right), \qquad \|f_\varepsilon - \langle f_\varepsilon \rangle\|_{L^2(t, x, y, v)} = O\left(\sqrt{\frac{\rho}{\sigma}}\right) \tag{5.18}$$

Therefore

$$\int_0^\infty dt \int_{\mathbb{R}} dx \left(\frac{\varepsilon^2}{\rho(\varepsilon)} \,|\langle\!\langle v_x \beta_x (f_\varepsilon - \langle f_\varepsilon \rangle) \rangle\!\rangle| \right)^2$$

$$\leqslant \int_0^T \int_{\mathbb{R}} \frac{\varepsilon^4}{\rho^2(\varepsilon)} \langle\!\langle (v_x\beta_x)^2 \rangle\!\rangle \langle\!\langle (f_\varepsilon - \langle f_\varepsilon \rangle)^2 \rangle\!\rangle \, dx\, dt \tag{5.19}$$

and by symmetry,

$$\int_0^\infty dt \int_{\mathbb{R}} dy \left(\frac{\varepsilon^2}{\rho(\varepsilon)} \,|\langle\!\langle v_y \beta_y (f_\varepsilon - \langle f_\varepsilon \rangle) \rangle\!\rangle| \right)^2$$

$$\leqslant \int_0^T \int_{\mathbb{R}} \frac{\varepsilon^4}{\rho^2(\varepsilon)} \langle\!\langle (v_y\beta_y)^2 \rangle\!\rangle \langle\!\langle (f_\varepsilon - \langle f_\varepsilon \rangle)^2 \rangle\!\rangle \, dt\, dy \tag{5.20}$$

Since

$$\langle\!\langle (v_x\beta_x)^2 \rangle\!\rangle = \frac{\varepsilon^2}{\sigma^2 \rho}, \qquad \text{and} \qquad \langle\!\langle (v_y\beta_y)^2 \rangle\!\rangle = \frac{\varepsilon^2}{\sigma^2 \rho}$$

The relations

$$\rho(\varepsilon) = O(\varepsilon^2), \qquad \sqrt{\rho(\varepsilon)} = O(\varepsilon)$$

imply

$$\frac{\varepsilon^2}{\rho(\varepsilon)} \langle\!\langle v.\beta.(z,v)(f_\varepsilon - \langle f_\varepsilon \rangle) \rangle\!\rangle$$

$$= \frac{\varepsilon^2}{\rho(\varepsilon)} \cdot \frac{\sqrt{\rho(\varepsilon)}}{\sigma(\varepsilon)} \to 0 \quad \text{and} \quad \varepsilon\, |\ln \sigma(\varepsilon)| \to 0, \quad \text{as } \varepsilon \to 0$$

Hence, to relate σ to ε, we must choose the function $\sigma(\varepsilon)$ so as to satisfy

$$\varepsilon\, |\ln \sigma| \to 0, \qquad \varepsilon^4 = O(\rho\sigma^2), \qquad \frac{\varepsilon^2}{\rho(\varepsilon)} |\ln \sigma| \to \kappa, \qquad \text{as} \quad \varepsilon \to 0 \tag{5.21}$$

Combining these relations leads to

$$\rho(\varepsilon) = O(\varepsilon), \qquad \varepsilon^2 (\ln \sigma) \sim C\rho, \qquad \varepsilon^4 = O(\rho\sigma^2) \tag{5.22}$$

Besides, Eqs. (5.21) can be rewritten as

$$\varepsilon\,|\ln\sigma| = O(1), \qquad \varepsilon^2 = O(\sigma^2\,|\ln\sigma|), \qquad \rho(\varepsilon) = O(\varepsilon) \tag{5.23}$$

From this, it follows easily that

$$\rho(\varepsilon) \sim \varepsilon^2 \ln\sigma(\varepsilon) \tag{5.24}$$

Put $\alpha(\varepsilon) = \sqrt{\sigma(\varepsilon)}$ and $\sigma(\varepsilon) = \varepsilon\,|\ln\varepsilon|^{\alpha}$; then, as $\varepsilon \to 0$, we have

$$|\ln(\sigma(\varepsilon))| \sim |\ln\varepsilon| \tag{5.25}$$

In order to satisfy (5.23), we must have $\alpha < 0$. Hence, if there exists $\sigma(\varepsilon)$ satisfying (5.23), we necessarily have

$$\rho(\varepsilon) \sim \varepsilon^2\,|\ln\sigma(\varepsilon)|, \qquad \text{as} \quad \varepsilon \to 0$$

which completes the choice of $\sigma(\varepsilon)$ and $\rho(\varepsilon)$.

Taking the Limit. Write the second term of the continuity equation in the form

$$\frac{\varepsilon^2}{\rho(\varepsilon)} \langle\!\langle v_x \beta_x(z, v) \langle f_\varepsilon\rangle(t, x, y, z)\rangle\!\rangle = \frac{\varepsilon^2}{\rho(\varepsilon)} \langle v_x \beta_x^\sigma\rangle(z) \langle f_\varepsilon\rangle(t, x, y, z) \tag{5.26}$$

Taking into account Lemma 4.1, we have:

$$\frac{\varepsilon^2}{\rho(\varepsilon)} \langle v_x \beta_x^\sigma\rangle(z) \rightharpoonup C\tilde{\psi}_\sigma(z) \qquad \text{in } L^\infty([0, h]) \tag{5.27}$$

where

$$C = \lim_{\varepsilon \to 0} \left[\frac{\varepsilon^2}{\rho(\varepsilon)}\,|\ln\sigma| \cdot \int_{\mathbb{R}} v_x^2 e^{-v_x^2/2} \frac{dv_x}{\sqrt{2\pi}} \right] \tag{5.28}$$

and

$$\tilde{\psi}_\sigma(z) = \int_{\mathbb{R}} \psi_\sigma(z, v_z)\, e^{-v_z^2/2} \frac{dv_z}{\sqrt{2\pi}} \tag{5.29}$$

Besides, $(1/|\ln\sigma|)\,\tilde{\psi}_\sigma(z)$ is bounded in $L^\infty([0, h])$, $(1/|\ln\sigma|)\,\tilde{\psi}_\sigma(z) \rightharpoonup \tilde{\psi}_\sigma(z)$ in w-$L^\infty([0, h])$ and $\langle f_\varepsilon\rangle(t, x, y, z) \rightharpoonup \langle f\rangle(t, x, y, z)$ in w-L^∞. Let $\chi(t, x, y)$ be a test function independent of z. We have

$$\frac{v_z}{2\pi}\,\partial_z \int_0^{+\infty}\int_{\mathbb{R}^4} \chi(t,x,y)\, f_\varepsilon(t,x,y,z,v)\, e^{-(v_x^2+v_y^2)/2}\, dt\, dx\, dy\, dv_x\, dv_y$$

$$= -\frac{1}{2\pi}\int_0^{+\infty}\int_{\mathbb{R}^4} \chi(t,x,y)(\partial_t + v_x\partial_x + v_y\partial_y)\, f_\varepsilon e^{-(v_x^2+v_y^2)/2}\, dt\, dx\, dy\, dv_x\, dv_y$$

$$-\frac{1}{2\pi}\int_0^{+\infty}\int_{\mathbb{R}^4} \chi(t,x,y)\,\sigma(f_\varepsilon - \langle f_\varepsilon\rangle)\, e^{-(v_x^2+v_y^2)/2}\, dt\, dx\, dy\, dv_x\, dv_y \qquad (5.30)$$

Now, set

$$F_\varepsilon = \int_0^{+\infty}\int_{\mathbb{R}^4} \chi(t,x,y)\, f_\varepsilon(t,x,y,z,v)\, G(v_x,v_y)\, dt\, dx\, dy\, dv_x\, dv_y \qquad (5.31)$$

where G being a Gaussian. The computation of $v_z\partial_z F_\varepsilon$ gives:

$$v_z\partial_z F_\varepsilon = \int_0^{+\infty}\int_{\mathbb{R}^2} dt\, dx\, dy\, v_z\partial_z\chi(t,x,y)\int_{\mathbb{R}^2} f_\varepsilon G(v_x,v_y)\, dv_x\, dv_y$$

$$+\int_0^{+\infty}\int_{\mathbb{R}^2} dt\, dx\, dy \int_{\mathbb{R}^2} G(v_x,v_y)\, dv_x\, dv_y$$

$$\times\chi(t,x,y)[-\rho(\varepsilon)\,\partial_t f_\varepsilon - \varepsilon v_x\partial_x f_\varepsilon - \varepsilon v_y\partial_y f_\varepsilon - \sigma(f_\varepsilon - \langle f_\varepsilon\rangle)] \qquad (5.32)$$

Integrating by parts the second term of (5.32) leads to

$$\int_0^{+\infty}\int_{\mathbb{R}^2} dt\, dx\, dy \int_{\mathbb{R}^2} G(v_x,v_y)\, dv_x\, dv_y\, \chi(t,x,y)$$

$$\times[-\rho(\varepsilon)\,\partial_t f_\varepsilon - \varepsilon(v_x\partial_x + v_y\partial_y)\, f_\varepsilon - \sigma(f_\varepsilon - \langle f_\varepsilon\rangle)]$$

$$= \int_{\mathbb{R}^2} G(v_x,v_y)\, dv_x\, dv_y \int_0^{+\infty}\int_{\mathbb{R}^2} dt\, dx\, dy$$

$$\times[\rho(\varepsilon)\,\partial_t\chi + (\varepsilon\,\partial_x\chi + \varepsilon\,\partial_y\chi)\, f_\varepsilon - \chi\sigma(f_\varepsilon - \langle f_\varepsilon\rangle)] \qquad (5.33)$$

Since

$$f_\varepsilon \in L^\infty, \qquad \partial_t\chi \in L^\infty, \qquad \partial_x\chi \in L^\infty, \qquad \partial_y\chi \in L^\infty \qquad (5.34)$$

we have

$$\|\rho(\varepsilon)\,\partial_t\chi + \varepsilon((\partial_x + \partial_y)\,\chi)\, f_\varepsilon\|_{L^\infty} = O(1) \qquad (5.35)$$

as well as $\partial_z \chi(t, x, y) \in \mathscr{C}_c^\infty$ and

$$\left\| \int_0^{+\infty} \int_{\mathbb{R}^2} dt\, dx\, dy\, v_z \partial_z \chi(t, x, y) \int_{\mathbb{R}^2} f_\varepsilon G(v_x, v_y)\, dv_x\, dv_y \right\|_{L^\infty(]0, h[\times \mathbb{R})} = O(1) \tag{5.36}$$

Besides, the last term of (5.33) gives

$$\|\sigma(f_\varepsilon - \langle f_\varepsilon \rangle)\|_{L^2(\mathbb{R}^+ \times \mathbb{R} \times \mathbb{R}^3)} = O(\sqrt{\rho\sigma}) \tag{5.37}$$

Finally we deduce that

$$v_z \partial_z F_\varepsilon \text{ is bounded in } L^\infty([0, h] \times \mathbb{R}_{v_z}) \qquad \text{and}$$
$$F_\varepsilon \text{ is bounded in } L^\infty([0, h] \times \mathbb{R}_{v_z}) \tag{5.38}$$

Then, using the Velocity Averaging Lemma (see ref. 16) we get that

$$\int_{\mathbb{R}} F_\varepsilon e^{-v_z^2/2} \frac{dv_z}{\sqrt{2\pi}} \qquad \text{converge uniformly to}$$
$$\int_0^{+\infty} \int_{\mathbb{R}^2} \chi(t, x, y) \langle f(t, x, y) \rangle\, dt\, dx\, dy \tag{5.39}$$

which completes the part of taking the limit.

To prove the last part of Theorem 2.2, integrating Eq. (2.12) over the variable t we obtain

$$\begin{aligned} &v_z \partial_z \int_0^t f_\varepsilon\, ds + \sigma \int_0^t (f_\varepsilon - \Pi f_\varepsilon)\, ds \\ &\quad = -\rho(\varepsilon)(f_\varepsilon(t, ., ., ., ., ., .) - f_0(t, .)) - \varepsilon \int_0^t v_x \partial_x f_\varepsilon\, ds - \varepsilon \int_0^t v_y \partial_y f_\varepsilon\, ds \end{aligned} \tag{5.40}$$

Since the problem (2.12)–(2.13) is invariant by translation, the relation (5.12) implies

$$\begin{aligned} &\|\partial_x f_\varepsilon\|_{L^\infty(\mathbb{R}^+ \times \mathbb{R}^2 \times [0, h] \times \mathbb{R}^3)} + \|\partial_y f_\varepsilon\|_{L^\infty(\mathbb{R}^+ \times \mathbb{R}^2 \times [0, h] \times \mathbb{R}^3)} \\ &\quad \leqslant e^{(\sigma(\varepsilon)/\rho(\varepsilon))\, \|B\|\, t} (\|\partial_x f_0\|_{L^\infty} + \|\partial_y f_0\|_{L^\infty}) \end{aligned} \tag{5.41}$$

From the estimate

$$\|\sigma(f_\varepsilon - \Pi f_\varepsilon)\|_{L^\infty(\mathbb{R}^+ \times \mathbb{R}^2 \times [0,h] \times \mathbb{R}^3)} \leqslant C\varepsilon^\alpha \tag{5.42}$$

the relation (5.40) gives

$$\left\| v_z \partial_z \int f_\varepsilon \, ds + \sigma \int_0^t (f_\varepsilon - \Pi f_\varepsilon) \, ds \right\|_{L^\infty(\mathbb{R}^+ \times \mathbb{R}^2 \times [0,h] \times \mathbb{R}^2)} = O(\varepsilon) \tag{5.43}$$

uniformly in $t \in [0, T]$. Applying the Velocity Averaging Theorem in dimension 1, i.e., Lemma 7 of ref. 16 leads to

$$\left\langle \left| \int_0^t f_\varepsilon(s, x, y, z', ., ., .,) \, ds - \int_0^t f_\varepsilon(s, x, y, z, ., ., .,) \, ds \right| \right\rangle = O(\varepsilon \, |\ln \varepsilon|) \tag{5.44}$$

uniformly in $t > 0$, $x \in \mathbb{R}$, $y \in \mathbb{R}$ and $z, z' \in [0, h]$.

Now start with the observation that if μ is a positive measure such that $\mu(x) = 1$, we have

$$\int \left(f - \int f \, d\mu \right)^2 d\mu = \int f^2 \, d\mu - 2 \int f \, d\mu \int f \, d\mu + \int d\mu \left(\int f \, d\mu \right)^2$$
$$= \int f^2 \, d\mu - \left(\int f \, d\mu \right)^2$$

Multiplying Eq. (2.12) by f_ε and integrating over all variables and figurating the sign, leads to

$$\frac{1}{2} \rho(\varepsilon) \, \partial_t \iint_{\mathbb{R}^3 \times D_h} f_\varepsilon{}^2 M \, dv \, dx \, dy \, dz + 0 + \left[\int_{\mathbb{R}^3} f_\varepsilon^2 v_z M \, dv_z \, dx \, dy \right]_0^h$$
$$+ \iint_{\mathbb{R}^3 \times D_h} \sigma(f_\varepsilon{}^2 - \langle f_\varepsilon{}^2 \rangle) \, M \, dv \, dx \, dy \, dz = 0 \tag{5.45}$$

Set

$$f_\varepsilon(x, y, 0, v)_{|v_z > 0} = \frac{\int_{v_z < 0} f_\varepsilon(x, y, 0, v) \, |v_z| \, M \, dv}{\int_{v_z > 0} v_z M \, dv} = \langle f_\varepsilon \rangle_-$$

and

$$f_\varepsilon(x, y, 0, v)_{|v_z < 0} = \langle f_\varepsilon \rangle_+$$

At the origin, write

$$-\tfrac{1}{2}\iint_{\mathbb{R}^3\times\mathbb{R}^2}(f_\varepsilon^2-\langle f_\varepsilon\rangle_-^2)\,|v_z|\,M\,dv\,dx\,dy=-\tfrac{1}{2}C_0\int_{\mathbb{R}^2}dx\,dy\langle f_\varepsilon^2-\langle f_\varepsilon\rangle_-^2\rangle$$

where

$$C_0=\int_{v_z<0}|v_z|\,M\,dv$$

and $\langle f_\varepsilon\rangle_-^2$ is a probability. But

$$-\tfrac{1}{2}\iint_{\mathbb{R}^3\times\mathbb{R}^2}(f_\varepsilon^2-\langle f_\varepsilon\rangle_-^2)\,|v_z|\,M\,dv\,dx\,dy$$

$$=-\tfrac{1}{2}C\int_{\mathbb{R}^2}dx\,dy\langle(f_\varepsilon-\langle f_\varepsilon\rangle_-)^2\rangle_-$$

where C is some constant.

On the other hand,

$$\int_0^T\tfrac{1}{2}\left(\int_{v_z<0}|v_z|\,M\,dv\right)\left[\int_{\mathbb{R}^2}dx\,dy\langle(f_{\varepsilon|_{z=h}}-\langle f_\varepsilon\rangle_{+|_{z=h}})^2\rangle_+\right.$$

$$\left.+\int_{\mathbb{R}^2}dx\,dy\langle(f_{\varepsilon|_{z=h}}-\langle f_\varepsilon\rangle_{-|_{z=h}})^2\rangle_-\right]dt$$

$$\leqslant\tfrac{1}{2}\rho(\varepsilon)\iint_{\mathbb{R}^3\times D_h}f_\varepsilon^2(0,x,y,z,v)\,M\,dv\,dx\,dy\,dz$$

$$=C_0\,\rho(\varepsilon)\tag{5.46}$$

Now, note that

$$C_0\int_0^T dt\int_{\mathbb{R}^2}dx\,dy(\langle(f_{\varepsilon|_{z=h}}-\langle f_\varepsilon\rangle_{+|_{z=h}})^2\rangle_++\langle(f_{\varepsilon|_{z=0}}-\langle f_\varepsilon\rangle_{-|_{z=0}})^2\rangle_-)$$

$$\leqslant C\rho(\varepsilon)\tag{5.47}$$

Let

$$g_\varepsilon=\int_0^t f_\varepsilon\,ds=O(1)\qquad\text{in } L^\infty([0,T])\tag{5.48}$$

Therefore

$$v_z \partial_z g_\varepsilon + \sigma(I - \Pi)\, g_\varepsilon = -\rho(\varepsilon)(f_\varepsilon(t) - f_\varepsilon(0)) - \varepsilon v_x \nabla_x g_\varepsilon$$
$$= O(\varepsilon) \qquad \text{in } L^\infty([0, T])$$

and

$$\langle |g_\varepsilon(z) - g(z')| \rangle = O(\varepsilon\, |\ln \varepsilon|)$$

uniformly in $t \in [0, T]$, x, $y \in \mathbb{R}$, v_x, $v_y \in \mathbb{R}$, z, $z' \in [0, h]$.

Now, observe that

$$\begin{aligned} &\int_{\mathbb{R}^2} \langle\!\langle |g_\varepsilon(z) - \langle\!\langle g_\varepsilon(z) \rangle\!\rangle| \rangle\!\rangle \, dx\, dy \\ &\quad \leqslant \int_{\mathbb{R}^2} \langle\!\langle |g_\varepsilon(z) - g_\varepsilon(0) \rangle\!\rangle| \rangle\!\rangle \, dx\, dy \\ &\qquad + \int_{\mathbb{R}^2} \langle\!\langle |g_\varepsilon(0) - \langle\!\langle g_\varepsilon(0) \rangle\!\rangle| \rangle\!\rangle \, dx\, dy \\ &\qquad + \int_{\mathbb{R}^2} \langle\!\langle |\langle\!\langle g_\varepsilon(0) \rangle\!\rangle - \langle\!\langle g_\varepsilon(z) \rangle\!\rangle| \rangle\!\rangle \, dx\, dy \end{aligned} \tag{5.49}$$

On the other hand, we have

$$\begin{aligned} &\int_{\mathbb{R}^2} \langle\!\langle |g_\varepsilon(z) - g_\varepsilon(0)| \rangle\!\rangle \, dx\, dy \\ &\quad + \int_{\mathbb{R}^2} \langle\!\langle |\langle g_\varepsilon(0) \rangle\!\rangle - \langle\!\langle g_\varepsilon(z) \rangle\!\rangle| \rangle\!\rangle \, dx\, dy = 2O(\varepsilon\, |\ln \varepsilon|) \end{aligned} \tag{5.50}$$

and

$$\begin{aligned} &\int_{\mathbb{R}^2} \langle\!\langle |g_\varepsilon(0) - \langle\!\langle g_\varepsilon(0) \rangle\!\rangle| \rangle\!\rangle \, dx\, dy \\ &\quad = \int_{\mathbb{R}^2} \langle\!\langle |g_\varepsilon(0) - \tfrac{1}{2} \langle\!\langle g_\varepsilon(0) \rangle\!\rangle_- - \tfrac{1}{2} \langle\!\langle g_\varepsilon(0) \rangle\!\rangle_+ | \rangle\!\rangle \, dx\, dy \\ &\quad = \int_{\mathbb{R}^2} \langle\!\langle |g_\varepsilon(0) - \tfrac{1}{2} \langle\!\langle g_\varepsilon(0) \rangle\!\rangle_- | \rangle\!\rangle_- \, dx\, dy \\ &\quad = O(\rho(\varepsilon)^{1/2}\, |\ln(\rho(\varepsilon))|^{1/2}) \end{aligned} \tag{5.51}$$

Combining (5.44)–(5.50) and (5.51) we thus obtain

$$\int_{\mathbb{R}^2} \langle\langle \, | g_\varepsilon(z) - \langle\langle g_\varepsilon(z) \rangle\rangle | \, \rangle\rangle \, dx\, dy = O(\varepsilon \, |\ln \varepsilon|) + O(\rho(\varepsilon)^{1/2} \, |\ln(\rho(\varepsilon))|^{1/2})$$

as announced. This completes the proof of Theorem 2.2. ∎

APPENDIX

Throughout this article, many basic topological linear spaces are utilized. Some of our notation regarding these spaces is standard while some of it is less so. These spaces, as well as our notation for them, are described below. A comprehensive treatment of them can be found in many references, for example ref. 14.

Let E be any normed linear space; $\|\cdot\|_E$ denotes its norm and E^* denotes its dual space. We shall use the notation w-E to indicate the space E equipped with its weak topology, that is the coarsest topology on E for which each of the linear forms

$$u \mapsto \langle w; u \rangle_{E^*, E} \qquad \text{for } w \in E^*$$

is continuous. Here $\langle \cdot; \cdot \rangle_{E^*, E}$ is the natural bilinear form relating E^* and E.

Let $(X, \mathscr{M}, dm)$ be a measure space and E a normed linear space. For every $1 \leqslant p \leqslant \infty$, we shall use the abbreviated notation $L^p(dm; E)$ for Bochner space $L^p((X, \mathscr{M}, dm); E)$ whenever there is no danger of confusion; we shall also use $L^p(dm)$ to denote the same space whenever E is a power of $\mathbb{R}$.

When Y is locally compact and dm is a Borel measure, we shall denote by $L^p_{\text{loc}}(dm; E)$ (or $L^p_{\text{loc}}(dm)$) the space determined by the family of seminorms

$$u \mapsto \left(\int_K \|u(y)\|_E^p \, dm(y) \right)^{1/p} \qquad \text{for compact} \quad K \subset Y$$

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